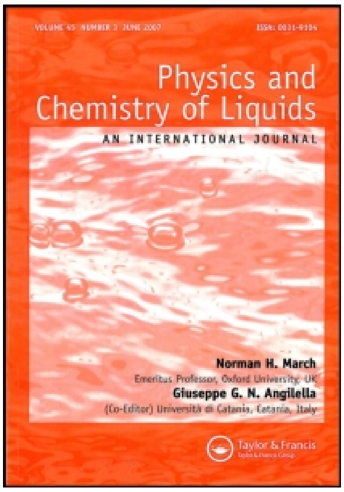

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Three-dimensional dilute Bose liquid at finite temperature: a Renormalization Group approach

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Three-dimensional dilute Bose liquid at finite temperature: a Renormalization Group approach

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We consider the influence of temperature on the critical behavior of a weakly interacting three dimensional Bose system. Using the flow equations of the Renormalization Group and a Φ^4 model with dynamical critical exponent $z=2$, we calculated the critical exponent ν, and the thermodynamical parameters near the critical temperature, taking in consideration the quantum effects. The quantum effects considered using this method gives for ν the value 0.75, which is close to the value 0.73 obtained by the polynomial method. The critical temperature shift ΔT_c was obtained, in the lowest-order approximation, and turns out to be proportional to the scattering length.

Keywords: Bose fluids; Critical exponent; Renormalization group

1. Introduction

The recent experimental data of the Bose–Einstein condensation (BEC) in magnetically trapped atomic vapors [1–3] give rise to a great interest for the theory of interacting bosonic systems. In dilute or weakly interacting Bose systems, the effect of the repulsive interaction has been studied by different methods [4–10]. The Gross–Pitaevskii [11] mean-field theory for condensate wave function has successfully described the condensate phase. The Renormalization Group (RG) method [8,9] has been applied to describe the critical behavior of the BEC, which was considered as a second-order phase transition, the weak repulsive interaction changing the universality class from Gaussian to that of XY-model. The simple model of a homogeneous interacting Bose gas has been treated by the RG method at $T=0$ [9], and at finite temperature T [8], and the results are in agreement to the self-consistent t-matrix theory [6]. As is well known, the critical behavior is determined by the dimensionality d of the system. In $d<2$ systems the RG theory is universal [7], but in the $d=2$ system the interacting

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Bose system presents the "quasicondensation", an effect similar to the BEC. However, in this case, near the critical temperature there is no universality, as was predicted by the t-matrix theory, and showed by RG method [8,12].

The Bose system in $d=3$ has been first treated using the RG method by Bijlsma and Stoof [13], the basic result of this study being a different critical behavior for the symmetric and broken-symmetry phases. Starting from a more realistic model, which takes into consideration the trapping effect, the Alber groups [14–17] used the RG method to study the critical behavior of the trapped interacting Bose gases. Using an energy-shell procedure and the ε-expansion, they obtained a more general flow equation system than from the homogeneous case [12], but in the thermodynamic limit the results from the case of homogeneous case are re-obtained. An important result obtained [16] is that, using the ε-expansion, they showed that the discrepancy between the symmetric and symmetry-broken phases [13] is an artifact, due to an improper treatment of the infrared divergencies appearing at finite temperature. The flow equations obtained in that study are written for the case of $z=2$, and solved in the high-temperature limit, giving for the critical exponent ν the value 0.600 which is closer to the 0.67 a value obtained in experiments. However, the polynomial expansion gives for this critical exponent the value 0.73.

The aim of our study is to perform a calculation of this exponent at low temperatures using the scaling field method [18], which has been employed in the study of the quantum effects in [19]. This method will enable thermodynamical properties to be calculated in the low-temperature domain, even for a $d=3$ interacting bosonic system. In this way we can estimate the importance of the quantum effects in this transition, which we also expect to be important for the critical temperature, coherence length, and the specific heat. We anticipate the importance of this effect on the critical exponent ν, compared to the previous estimations.

The article is organized as follows: In section 2 we present the model and the flow equations. The scaling field and the thermodynamic properties will be evaluated in section 3. Using these results, we also calculate the critical condensed density of particles and the critical temperature. The results will be discussed in section 5, in connection with some results from fundamental principles.

2. Model and scaling equations

We start with the standard action [12,13]:

$$S=\frac{1}{2}\int \mathrm{d}\tau\, \mathrm{d}^3\mathbf{r}\Big[(\partial_\tau-\nabla^2+r_0)|\Phi|^2+\frac{u_0}{8}|\Phi|^4\Big], \tag{1}$$

where τ is the imaginary time, r_0 is the chemical potential, u_0 is the bare interaction, and $\Phi(\tau,\mathbf{r})$ is the n-component bosonic field. This action has been studied by RG method in the $k=(\mathbf{k},\omega_n)$ space by integrating out the fast degrees of freedom and rescaling the variables as $\mathbf{k}'=\mathbf{k}b$, $\omega_n'=b^z\omega_n$, $T'=b^zT$, and the field operator as $\Phi'(\mathbf{k}',\omega_n')=b^{-(d+z+2)}\Phi(\mathbf{k}/b,\omega_n/b^z)$, where z is the dynamical critical exponent, and the renormalization parameter b is parameterized as $b=\exp(l)$.

The scaling equations for the parameters $T(l)$, $r(l)$, and $u(l)$ have been obtained following the general method [8,12–14], as:

$$\frac{\mathrm{d}T(l)}{\mathrm{d}l} = zT(l), \tag{2}$$

$$\frac{\mathrm{d}r(l)}{\mathrm{d}l} = 2r(l) + \frac{n+2}{4}u(l)K_3F_1[r(l), T(l)], \tag{3}$$

$$\frac{\mathrm{d}u}{\mathrm{d}l} = 4 - (d+z)u(l) - \frac{n+8}{4}K_3F_2[r(l), T(l)]. \tag{4}$$

In these equations the function $F_1[r(l), (T(l)]$ has the form:

$$F_1[r(l), T(l)] = \coth\frac{1+r(l)}{2T(l)}, \tag{5}$$

which is different from [8,12], but identical with [16]. The difference is in fact a constant, and it is connected to the presence of the two channels in the perturbation calculation. The function $F_2[r(l), T(l)]$ is given in [8,12], but in the low-temperatures domain it is taken as a constant, and in this way we can solve the equation for $u(l)$. In this case, the expansion parameter is $\varepsilon = 2 - d$, and for $d=3$ it is negative. We will perform the calculations using the notation $d = 2 + |\varepsilon|$. Near the quantum critical point (QCP) we can solve the equation for $u(l)$ exactly and the solution is:

$$u(l) = \frac{u_0 e^{-|\varepsilon| l}}{1 + [(K_3/4)e^{-|\varepsilon| l} - 1]}, \tag{6}$$

where $K_3 = 1/2\pi^2$. Following the method developed in [19], we solve the equation for $r(l)$ and the solution has the form:

$$r(l) = e^{2l}\left[r_0 - r_{0c} + \frac{1}{2}K_3Tu(l)\ln\frac{1}{1+e^{-1/T(l)}}\right] - \frac{K_3u(l)}{4} \tag{7}$$

where $r_{0c} = K_3u_0/4$.

These solutions will be used in the next section to study the influence of the quantum effects on the critical behavior. The general method applied to solve this problem is to obtain the solutions of the flow equations (2–4) and to get the scaling fields, which will be used to calculate the thermodynamic quantities.

However, in the critical region all these quantities are singular. We can study the influence of quantum effects on the critical behavior taking a value of l on the flow equations which is close to the critical region, but in such a way that we can apply the perturbational methods. This value, denoted by l^* will be calculated in the next section.

3. Scaling field and thermodynamics

The scaling field $t_r(l)$ is defined as [18,19]:

$$t_r(l) = r(l) + \frac{K_3 u(l)}{4}, \tag{8}$$

and using equation (6) we get for the scaling field the general form $t_r(l) = e^{2l} t_r(T)$, where:

$$t_r(T) = r_0 - r_{0c} + \frac{K_3 u(l) T}{2} \ln \frac{1}{1 + \exp[-1/T(l)]}. \tag{9}$$

In order to describe the critical behavior, we will calculate the thermodynamic parameters in the symmetric phase using the scaling field $t_r(l)$ as in [18] for classical phase transitions and [19] for the quantum case. The basic idea is to get an expression for $l^\star$ which drives us out of the critical region but still contains the influence of the quantum criticality. We get it using the condition $t_r(l^\star) = 1$. The obtained value of $l^\star$ is necessary in the calculations of the thermodynamic quantities in the critical region. From this condition we get for $T(l^\star) \simeq 1/u_0$, and the expression for $l^\star$ is approximated by the expression:

$$l^\star \simeq -\frac{1}{2} \ln[T/C(u_0)], \tag{10}$$

where $C(u_0) = (K_3 u_0/2) \ln 1/u_0$. The temperature dependence of $t_r(T)$ in the critical region will be obtained from equation (8) using equation (6) for $u(l^\star)$. In this way we get, in the critical domain using equations (9) and (10), the expression:

$$t_r(T) \simeq r_0 - r_{0c} + \frac{K_3 u_0 \ln(1/u_0)}{2} T^{3/2} \tag{11}$$

This is one of the most important result of our calculation, because the coherence length $\xi(T)$ is given [19]:

$$\xi(T) \sim [t_r(T)]^{-1/2}, \tag{12}$$

and the critical exponent ν, defined by $\xi(T) \sim T^{-\nu}$, is 0.75, very close to the value 0.73 obtained by the polynomial expansion [16]. An important quantity of this system is the density of the particles in the critical region given by [12]:

$$n = \frac{1}{4\pi} T^{3/2} \ln\left[1 - \exp\left(\frac{r(l^\star)}{T(l^\star)}\right)\right] \tag{13}$$

We can calculate the ratio $r(l^\star)/T(l^\star)$ from $T(l^\star) = Te^{2l^\star}$ and the relation (7), as:

$$\frac{r(l^\star)}{T(l^\star)} \simeq T^{\varepsilon/2} u_0 I_{3D} \tag{14}$$

where I_{3D} is:

$$I_{3D} = \frac{1}{2}\int_{e^{-2l^\star}/T}^{1/T} \mathrm{d}x \frac{x^{|\varepsilon|/2}}{e^x - 1}, \tag{15}$$

which in the low-temperature domain ($T \longrightarrow 0, l^\star \gg 1$) can be calculated exactly as: $I_{3D} = (1/2)\Gamma(1/2)\zeta(1/2)$, where $\Gamma(m)$ is the gamma function and $\zeta(n)$ is the Riemann Zeta. From equations (13)–(15) we obtain for the density n the expression:

$$n = \frac{1}{4\pi} T^{3/2} \ln\left[1 - \exp(-aT^{1/2})\right], \tag{16}$$

where $u_0 = 8\pi a$ is the coupling constant and a is the s-wave scattering length. The density of the free energy f is given by the general equation [12]:

$$\frac{\mathrm{d}f}{\mathrm{d}l} = (d + z)f + g[r(l), T(l)], \tag{17}$$

where $f = f[r(l), T(l)]$. The general expression for $g[r(l), T(l)]$ has the form:

$$g[r(l), T(l)] = K_d \sum_{\omega_n} [1 + r(l) - i\omega_n], \tag{18}$$

and performing the summation on the bosonic Matsubara frequency ω_n we get:

$$g[r(l), T(l)] = K_d T \ln\left[\exp\frac{1 + r(l)}{T} - 1\right] - \frac{K_d}{2}[1 + r(l)]. \tag{19}$$

Using this result, we calculate the free energy density $f(r_0, T)$ as:

$$f(r_0, T) = f_0 + \frac{K_d}{2}\int_0^{l^\star} \mathrm{d}l' e^{-(d+2)l'}[1 + r(l')], \tag{20}$$

where f_0 is the free energy density in the absence of fluctuations. For the case $d = 3$ system, we will calculate the free energy density $f(r_0, T)$ using for $r(l')$ given by equation (7) the expression:

$$r(l') \simeq -\frac{K_3 T \exp(2l')}{2} \ln[1 + e^{-1/T(l')}]. \tag{21}$$

From equation (20) using equations (21) and (10) we obtain for the free energy density:

$$f(T) \simeq f_0 - CT^{2+(3/2)}, \tag{22}$$

where C is a constant. The specific heat $c(T) = -T\partial^2 f/\partial T^2$ will be calculated near the critical point as:

$$c(T)/T \sim T^{3/2} \tag{23}$$

which is a particular case of the scaling behavior which gives for the critical exponent $\alpha = -(d-2)/2$.

4. Critical temperature

The interaction dependence of the critical temperature T_{c} of dilute homogeneous Bose gas has a long and controversial history, and the different results of the critical temperature shift $\Delta T_{\mathrm{c}} = (T_{\mathrm{c}} - T_{\mathrm{c}0})/T_{\mathrm{c}0}$ ($T_{\mathrm{c}0}$ is the critical temperature for the BEC of the free Bose gas) are presented in [20]. Using the model described by equation (1), many authors obtained that the critical temperature shift ΔT_{c} increases linearly with the scattering length a. The systematic investigation of the problem by the Baym group [21–25] leads to the dependence of the critical temperature shift $\Delta T_{\mathrm{c}} = can^{1/3}$. This result has been obtained in the lowest order, and the values obtained for the constant c are very different. The higher order corrections have been obtained [26], but even at the present time there are important differences between the numerical values [20]. In the following, we will use the equation (13) to calculate the dependence of the critical temperature of the condensed state [12]. The quantity $r(l^\star)/T(l^\star)$ is given by equation (14) which we will calculate as $I_{3D} = \Gamma(3/2)\zeta(3/2)$, taking the integration limits in equation (15) as zero and infinity. The equation for the critical temperature T_{c} becomes:

$$n = \frac{T_{\mathrm{c}}^{3/2}}{4\pi} \ln\left[1 - \exp(-acT_{\mathrm{c}}^{1/2})\right] \tag{24}$$

where $c = (\pi)^{-1/2}\zeta(3/2)$. In the right-hand side of this equation, we approximate $T_{\mathrm{c}} \simeq T_{\mathrm{c}0} = (4\pi)^{1/2}[n/\zeta(3/2)^{2/3}]$. From equation (24), in the approximation of small T_{c} we get:

$$T_{\mathrm{c}} \simeq T_{\mathrm{c}0}\left[1 + 2.53an^{1/3}\right] \tag{25}$$

Equation (25) gives $\Delta T_{\mathrm{c}}/T_{\mathrm{c}0} = can^{1/3}$, where $c \simeq 2.53$ can be compared with $c = 2.5$ [22,23] obtained by ladder summation. The dependence of the ratio $T_{\mathrm{c}}/T_{\mathrm{c}0}$ as function of the parameter $n^{1/3}a$ is given in figure 1. This linear increase is relevant in the approximation of dilute interacting Bose gas.

Following the general method applied recently by March and Tosi [27] for the quasi-two-dimensional bosonic system, we can calculate the localization length l_{c}

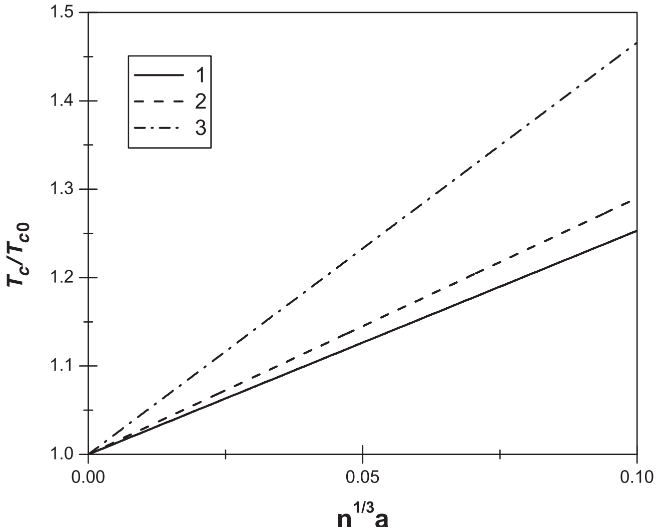

Figure 1. The ratio T_c/T_{c0} plotted as a function of the parameter $n^{1/3}a$, for different values of c. Line 1 was obtained using $c = 2.53$, line 2 corresponds to $c = 2.9$ [22] and line 3 corresponds to $c = 4.66$ [13].

from $\epsilon \sim l_c^{-2}$ using equation (24), in the approximation of a weak interaction. As in the case of the quasi-two-dimensional systems we get $l_c \sim n^{-1/2}$.

5. Discussion

In this section we will discuss our main result, the value of the critical exponent $\nu = 0.75$, in connection with the existent values calculated in literature. The value $\nu = 0.600$ is obtained in the lowest order [16], which is the same for the case of the classical Wilson theory which is due to the approximation of the Bose function $f_B(E)$ by $f_B(E) \simeq T/(E - \mu)$. We mention that the same approximation has been made in [13] and results give the critical exponent $\nu = 0.53$, which is close to 0.600, considered as a reference value. A better result was obtained in [16] $\nu = 0.650$ which improved the approximation, but the calculation still remains in the classical regime. The quantum effects have been captured by Andersen and Strickland [28], retaining RG calculations of quantum effects. In this way they got $\nu = 0.73$, a value which is close to our result, but still different to $\nu = 0.67$ [20,29] obtained for ^{4}He. We suspect that the inclusion of quantum effects in the RG calculations overestimated these effects compared to the classical fluctuation effects, but also some more accurate measurements on real dilute bosonic systems (^{4}He). We hope that more accurate measurements on other thermodynamic quantities in the critical region on the atomic vapors will provide useful information on other critical exponents. The dependence of the critical temperature T_c on $n^{1/3}a$ has been measured recently [30] and our results are in agreement with these data in the low-temperature regime. As was recently remarked in [25], the linear dependence of the critical temperature shift is essentially connected to the approximation of the Bose function by the classical expression used in [16] for the calculation of the critical exponent ν, but it gives also different values for the constant c. From figure 1 we can see that our result is close to the case from [22], where the quantum effects are considered, and quite different as magnitude from [13] where the classical approximation was used.

The calculation of the critical temperature shift in higher order in a is still an open problem.

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